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ARTICLE

Shooting Method for the Solution of Nonlinear Boundary Value Problems

Grace O. Akinlabi[1]*, George S. Nduka[2]

[1]Department of Mathematics, Covenant University, Ota, Nigeria
[2]Department of Mathematics, Dennis Osadebay University, Asaba, Nigeria

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**Corresponding author*
Grace O. Akinlabi E-mail: grace.akinlabi@covenantuniversity.edu.ng; +2347034644611

Abstract
This work describes the shooting method for the solution of a second-order nonlinear Boundary Value Problem (BVP). This method works by first transforming each BVP into a system of Initial Value Problems (IVPs). The initial conditions associated with the IVPs are then adjusted to match the boundary conditions associated with the BVPs by making guesses or “shooting for values”. This process is repeated using the secant method to determine the right value until the initial conditions are satisfactorily closed to the boundary conditions. The IVPs are solved using the Euler method. The Euler method was chosen for this work primarily due to its simplicity and ease of implementation. An illustrative example is considered and the results obtained show the importance of the shooting method to BVPs.

Keywords: Boundary Value Problems, Non-linear Equations, Ordinary Differential Equations, Shooting Method

1.0 Introduction

The Shooting Method (SHM) is a numerical scheme for solving Boundary Value Problems (BVPs), typically by reducing the BVP to a system of Initial Value Problems (IVPs). This technique iteratively identifies a suitable initial condition for the IVP that matches the boundary condition of the original BVP. Solutions of BVPs using SHM are often derived using integration schemes such as the Euler and Runge-Kutta methods (Filipov, Gospodino, and Farago, 2017).

The motivation for employing the shooting method to solve nonlinear boundary value problems (BVPs) often stems from the necessity to address the complexities of such problems, which frequently arise in scientific and engineering applications. These include versatility in handling nonlinearity, accuracy, control, and efficiency for specific problem categories. In this study, we provide a detailed description of the SHM and its application in solving second-order BVPs. The shooting approach is highly effective and has been applied to solve projectile problems. It is a conventional method proposed for solving BVPs, and its performance largely depends on the stability of the associated boundary conditions (Gladwell, 2008).

The SHM and its modified forms have been utilized by several authors to solve BVPs. For example, Holsapple, Venkataraman, and Doman (2003) proposed a modified simple shooting technique for solving two-point BVPs. This method highlights the two aspects of SHM: the multiple shooting method (MSM) and the simple shooting method (SSM). In another study, Ahsan and Farrukh (2013) developed an iterative formula that converts a BVP into a system of IVPs, which are then solved using SHM and interpolation. They also compared solutions obtained using the SHM with those from other methods, such as the Euler and Runge-Kutta methods. Bailey and Shampine (1968) explored the theoretical similarities between discrete and continuous problems, noting that combining Newton's method with a suitable multiple shooting method and a method of continuation provides a highly efficient tool.
Another study tested the SHM in solving BVPs of second-order ordinary differential equations, revealing some inaccuracies in initial values and concluding that the shooting method is the most straightforward approach to solving BVPs (Adam and Hashim, 2014). However, the shortcomings of the shooting method, such as the increased complexity of the differential equations before the IVP can be appropriately integrated, have also been discussed. For example, Morrison, Riley, and Zancanaro (1962) illustrated the multiple shooting method using several examples. Attili and Syam (2008) also noted that the guarantee of the existence and uniqueness of solutions to BVPs is an important consideration.

The classical shooting method has been successfully used to verify the existence and multiplicity results for BVPs of second-order ODEs. It is argued that the cone method may be more reliable for higher-order equations and PDEs. However, the shooting method has the advantage of requiring only two solutions with similar properties. The Dirichlet boundary value problem for nonlinear equations was also examined, demonstrating the effectiveness of the shooting method (Kwong, 2006).
A convergent theorem was developed for the shooting method, combining the explicit Euler scheme and Newton's iteration technique for solving nonlinear two-point BVPs (Keller, 1968). The multiple shooting method (MSM) was shown to reduce the progression of the solution of IVPs by splitting an interval into several subintervals and concurrently adjusting the initial data to satisfy the initial conditions and ensure continuity (Davis, 1984). A boundary value solver based on the shooting method with error control yielded better performance than the linear shooting method (Burden, Faires, and Burden, 2015).

In solving the initial value problem derived from the boundary value problem, various numerical methods can be used, such as the Runge-Kutta 4th order method and Newton's method. The Taylor series method can also provide accurate results with minimal error. However, the Runge-Kutta method remains the most efficient as it does not require prior calculation of higher derivatives, unlike the Taylor method (Grewal, 2014). Additional research on shooting methods can be found in studies by Edun and Akinlabi (2021), Javeed, Shabnam, and Baleanu (2020), and Pellegrini and Russell (2020).

The BVPs to be considered in this work is of the form:

$$\begin{cases} y'' = f(x,\ y(x),\ y(x)') \\ y(a) = \alpha, \quad y(b) = \beta \end{cases} \text{ on the interval } [a,b] \qquad (1.1)$$

2.0 Methodology

2.1 Existence and Uniqueness Theorems of a Nonlinear Second Order BVP

Existence

Let the function $f(x,\ y,\ y')$ be continuous on an interval $[a,b]$ and satisfies certain growth conditions (such as those provided by the Schauder fixed-point theorem), then a solution to the nonlinear bvp below exist:

$$y'' = f(x,\ y,\ y') \qquad x \in [a,b]$$
$$y(a) = \alpha \quad , y(b) = \beta \qquad (2.1)$$

Uniqueness

Let the function $f(x,\ y,\ y')$ satisfies a Lipschitz condition with respect to y and y':

$$\left| f(x,\ y_1,\ y_1') - f(x,\ y_2,\ y_2') \right| \le L\left(\left| y_1 - y_2 \right| + \left| y_1' - y_2' \right| \right), \qquad (2.2)$$

where L is a Lipschitz constant, then the solution to the nonlinear BVP is unique.

2.2 The Solution Technique-Shooting Method

Let us consider a second-order BVP of the form:

$$\begin{cases} y'' = g(x,\ y(x),\ y(x)') \\ y(a) = \alpha, \quad y(b) = \beta \end{cases} \text{ on the interval } [a,b] \qquad (2.3)$$

where the initial condition is $y(a) = \alpha$ and the final condition is $y(b) = \beta$, which we will assume to be $y'(a) = v$. Then the resulting IVP will be

$$\begin{cases} y'' = g(x,\ y(x),\ y(x)') \\ y(a) = \alpha, \quad y'(a) = v \end{cases} \tag{2.4}$$

Let $\varphi(v) = \beta$, the aim is to adjust v until the value is close to β at a desired accuracy.

We start by making an initial guess for v, assuming it to be α_0; then we solve the resulting IVP presented in (2.5) via the Euler method.

$$\begin{cases} y'' = g(x,\ y(x),\ y(x)') \\ y(a) = \alpha, \quad y'(a) = \alpha_0 \end{cases} \tag{2.5}$$

This process is continued until we get a suitable value for v. That is, another guess is made for v, assuming it to be α_1; then we obtain the associated IVP presented in (2.6).

$$\begin{cases} y'' = g(x,\ y(x),\ y(x)') \\ y(a) = \alpha, \quad y'(a) = \alpha_1 \end{cases} \tag{2.6}$$

This process continues until we hit the target, β. In case this is impossible, we use the estimating interpolation formulation (2.7) for a better value, α_2 based on the first and second guesses: α_0 and α_1.

$$\frac{\alpha_2 - \alpha_1}{y(b) - \varphi(\alpha_1)} = \frac{\alpha_1 - \alpha_0}{\varphi(\alpha_1) - \varphi(\alpha_0)} \tag{2.7}$$

From which we obtain:

$$\alpha_2 = \alpha_1 + \left(\frac{\alpha_1 - \alpha_0}{\varphi(\alpha_1) - \varphi(\alpha_0)} \right) \left[y(b) - \varphi(\alpha_1) \right] \tag{2.8}$$

In general,

$$\alpha_{n+1} = \alpha_n + \left(\frac{\alpha_n - \alpha_{n-1}}{\varphi(\alpha_n) - \varphi(\alpha_{n-1})} \right) \left[y(b) - \varphi(\alpha_n) \right], \quad (n \geq 1) \tag{2.9}$$

The shooting method can be adopted in line with other semi-analytical methods for solving problems in engineering, finance, and applied sciences (Tian, Yuan, Li, Zhang, and Ghanbarnezhad-Moghanloo, 2024; Magani, Ogundile, and Edeki, 2022; Jalili, Sadeghi Ghahare, Jalili, and Domiri Ganji, 2023; Edeki, Jena, Chakraverty, and Baleanu, 2020; Chen, Hou, Chen, Song, Lin, Jin, and Chen, 2023; Akinlabi, Edeki, and Braimah, 2022).

3.0 Result and Discussion

In this section, two illustrative second-order nonlinear BVPs are solved to illustrate the shooting method described in the previous section. These BVPs are first converted to systems of IVPs and then solved via Euler's method application. The Secant method is also adopted in obtaining the best value for the initial condition. The solutions obtained are compared with the exact using tables and graphs.

Case Example: Consider the second-order nonlinear BVP:

$$\begin{cases} y'' = 2y^3, \quad x \in [1, 2] \\ y(1) = 0.25, \quad y(2) = 0.2 \end{cases} \tag{3.1}$$

With the exact solution: $y(x) = \dfrac{1}{x+3}$ (3.2)

Solution of the Case Example

The BVP in (3.1) is reduced to a system of IVPs as follows:

$$\begin{aligned} y_1' &= y_2, \quad y_1(1) = 0.25 \\ y_2' &= 2y_1^3, \quad y_2(1) = \alpha_0 \end{aligned} \tag{3.3}$$

For the first guess, let $\alpha_0 = 2$, and step size $h = 0.25$, using the Euler's method we have:

$$\begin{aligned} y_1' &= f_1(x, y_1, y_2) = y_2 \\ y_2' &= f_2(x, y_1, y_2) = 2y_1^3 \end{aligned} \tag{3.4}$$

$$\begin{aligned} y_{1,n+1} &= y_{1,n} + hf_1(x_n, y_{1,n}, y_{2,n}) = y_{1,n} + hy_{2,n} \\ y_{2,n+1} &= y_{2,n} + hf_2(x_n, y_{1,n}, y_{2,n}) = y_{2,n} + h(2y_{1,n}^3) \end{aligned} \tag{3.5}$$

When $n = 0$,

$$\begin{aligned} y_{1,1} &= y_{1,0} + hy_{2,0} = y_{1,0} + h\alpha_0 = 0.25 + 0.25(2) = 0.75 \\ y_{2,1} &= y_{2,0} + 2hy_{1,0}^3 = \alpha_0 + 2hy_{1,0}^3 = 2 + 2(0.25)(0.25)^3 = 2.00781 \end{aligned} \tag{3.6}$$

When $n = 1$,

$$\begin{aligned} y_{1,2} &= y_{1,1} + hy_{2,1} = 0.75 + 0.25(2.00781) = 1.25195 \\ y_{2,2} &= y_{2,1} + 2hy_{1,1}^3 = 2.00781 + 2(0.25)(0.75)^3 = 2.21875 \end{aligned} \tag{3.7}$$

When $n=2$,

$$y_{1,3}=y_{1,2}+hy_{2,2}=1.25195+0.25(2.21875)=1.80664$$
$$y_{2,3}=y_{2,2}+2hy_{1,2}^{3}=2.21875+2(0.25)(1.25195)^{3}=3.19989$$

(3.8)

When $n=3$,

$$y_{1,4}=y_{1,3}+hy_{2,3}=1.80664+0.25(3.19989)=2.60661$$

(3.9)

This implies that at $x_4=2,$ we have $y(2)=y_{1,4}=2.60661$.

But the target is $\beta=0.2$, hence we make another guess

$$y_1'=y_2, \quad y_1(1)=0.25$$
$$y_2'=2y_1^3, \quad y_2(1)=\alpha_1$$

(3.10)

For the second guess, let $\alpha_1=0$, and step size $h=0.25$, we have the following using the Euler's method :

When $n=0$,

$$y_{1,1}=y_{1,0}+hy_{2,0}=y_{1,0}+h\alpha_1=0.25+0.25(0)=0.25$$
$$y_{2,1}=y_{2,0}+2hy_{1,0}^{3}=\alpha_1+2hy_{1,0}^{3}=0+2(0.25)(0.25)^{3}=0.0078125$$

(3.10)

When $n=1$,

$$y_{1,2}=y_{1,1}+hy_{2,1}=0.25+0.25(0.0078125)=0.25195$$
$$y_{2,2}=y_{2,1}+2hy_{1,1}^{3}=0.0078125+2(0.25)(0.25)^{3}=0.015625$$

(3.11)

When $n=2$,

$$y_{1,3}=y_{1,2}+hy_{2,2}=0.25195+0.25(0.015625)=0.25586$$
$$y_{2,3}=y_{2,2}+2hy_{1,2}^{3}=0.015625+2(0.25)(0.25195)^{3}=0.02362$$

(3.12)

When $n=3$,

$$y_{1,4}=y_{1,3}+hy_{2,3}=0.25586+0.25(0.02362)=0.26177$$

(3.13)

This implies that at $x_4=2,$ we have $y(2)=y_{1,4}=0.26177$. But the target is $\beta=0.2$, hence we make another guess by using the secant method, we make a third initial guess, α_2.

$$\alpha_k=\alpha_{k-1}-\frac{\left(y(b,\alpha_{k-1})-\beta\right)\left(\alpha_{k-1}-\alpha_{k-2}\right)}{y(b,\alpha_{k-1})-y(b,\alpha_{k-1})}$$

(3.14)

$$\alpha_2=\alpha_1-\frac{(y(b,\alpha_1)-\beta)(\alpha_1-\alpha_0)}{y(b,\alpha_1)-y(b,\alpha_0)}=0-\frac{(0.26177-0.2)(0-2)}{0.26177-2.60661}=-0.05268\simeq-0.053$$

For the third guess, let $\alpha_2=-0.053$, and step size $h=0.25$, we have the following using the Euler's method :

When $n=0$,

$$y_{1,1}=y_{1,0}+hy_{2,0}=y_{1,0}+h\alpha_2=0.25+0.25(-0.053)=0.23675$$
$$y_{2,1}=y_{2,0}+2hy_{1,0}^{3}=\alpha_2+2hy_{1,0}^{3}=-0.053+2(0.25)(0.25)^{3}=-0.04519$$

When $n=1$,

$$y_{1,2}=y_{1,1}+hy_{2,1}=0.23675+0.25(-0.04519)=0.22545$$
$$y_{2,2}=y_{2,1}+2hy_{1,1}^{3}=-0.04519+2(0.25)(0.23675)^{3}=-0.03856$$

Akinlabi and Nduka *Shooting Method for the Solution of Nonlinear Boundary Value Problems*

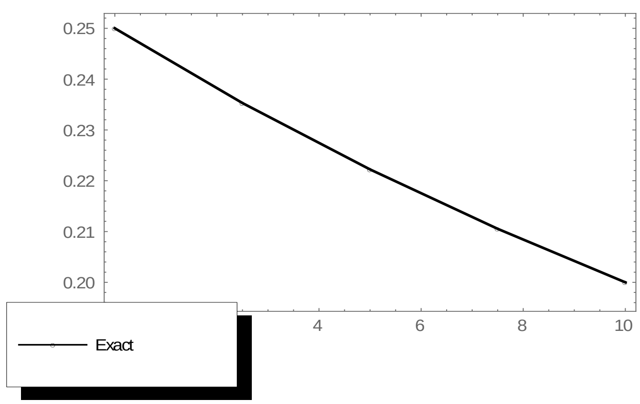
(3.15)

When $n=2$,

$$y_{1,3}=y_{1,2}+hy_{2,2}=0.22545+0.25(-0.03856)=0.21581$$
$$y_{2,3}=y_{2,2}+2hy_{1,2}^{3}=-0.03856+2(0.25)(0.22545)^{3}=-0.03283$$

Table 3.1: Comparison of the shooting method with the exact for the case example

i	x_i	SHM	Exact	$\lvert\varepsilon\rvert$
0	1	0.25000	0.25000	0.00000
1	1.25	0.23500	0.23529	2.9×10^{-4}
2	1.5	0.22195	0.22222	2.7×10^{-4}
3	1.75	0.21052	0.21053	1×10^{-5}
4	2	0.20001	0.20000	1×10^{-5}



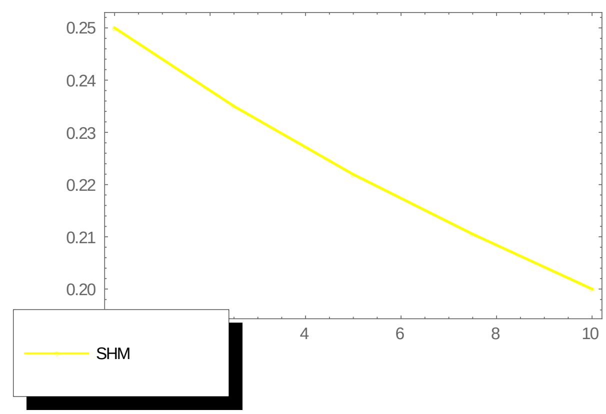


Figure 3.1: Graph comparing the shooting method and the exact

When $n=3$,

$$y_{1,4} = y_{1,3} + hy_{2,3} = 0.21581 + 0.25\left(-0.03283\right) = 0.2076 \quad (3.17)$$

This implies that at $x_4 = 2,$ we have $y(2) = y_{1,4} = 0.2076$. But the target is $\beta = 0.2$, hence we make another guess as the result obtained is not still satisfactory, we make the fourth guess,

$$\alpha_3 = \alpha_2 - \frac{(y(b,\alpha_2)-\beta)(\alpha_2-\alpha_1)}{y(b,\alpha_2)-y(b,\alpha_1)} \quad (3.18)$$

$$= -0.053 - \frac{(0.2076-0.2)(-0.053-0)}{0.2076-0.26177} = -0.06043 \simeq -0.06 \quad (3.19)$$

For the fourth guess, let $\alpha_3 = -0.06$, and step size $h = 0.25$, using the Euler's method we have:

When $n=0$,

$$y_{1,1} = y_{1,0} + hy_{2,0} = y_{1,0} + h\alpha_3 = 0.25 + 0.25(-0.06) = 0.235$$

When $n=1$,

$$y_{1,2} = y_{1,1} + hy_{2,1} = 0.235 + 0.25\left(-0.05219\right) = 0.22195$$

$$y_{2,2} = y_{2,1} + 2hy_{1,1}^{3} = -0.05219 + 2\left(0.25\right)\left(0.235\right)^{3} = -0.0475 \quad (3.20)$$

When $n = 2$,

$$y_{1,3} = y_{1,2} + hy_{2,2} = 0.22195 + 0.25\left(-0.0475\right) = 0.21052$$

$$y_{2,3} = y_{2,2} + 2hy_{1,2}^{3} = -0.0475 + 2\left(0.25\right)\left(0.22195\right)^{3} = -0.04203 \quad (3.21)$$

When $n = 3$,

$$y_{1,4} = y_{1,3} + hy_{2,3} = 0.21052 + 0.25\left(-0.04203\right) = 0.20001 \quad (3.22)$$

This implies that at $x_4 = 2,$ we have $y\left(2\right) = y_{1,4} = 0.20001$. We have obtained our satisfactory result since the target is $\beta = 0.2$.

The Table 1 and Figure 1 show the comparison of the solutions obtained using the Shooting method with the exact method.

4.0 Conclusion

The present work describes a shooting method for iteratively solving second-order nonlinear BVPs. The method entails converting the BVP into a set of IVPs, which are then solved with the Euler method. The secant method is used to iteratively transform the initial conditions until the solution meets the boundary conditions or requirements. The illustration case shows how well the shooting method works for solving nonlinear BVPs. This method enables accurate and efficient numerical solutions, making it particularly helpful in situations without analytical solutions. The reliability and applicability of this technique are demonstrated by the application of the secant method for refining initial guesses and the Euler method for solving the IVPs. In a variety of scientific and engineering ...d is a viable tool for dealing with BVPs. It is an essential component in the numerical analysis field because of its capacity to simplify complicated issues into more understandable forms. To further improve the flexibility and correctness of the shooting method in solving nonlinear BVPs, future research needs to explore more advanced numerical methods for solving the IVPs and different approaches for refining initial assumptions.

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Conflict of Interest

No conflict of interest.

Authors Contribution

Conception: G.O. Akinlabi
Design: G.O. Akinlabi, G.S Nduka
Execution: G.O. Akinlabi
Interpretation: G.O. Akinlabi
Writing the paper: G.O. Akinlabi, G.S Nduka

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